Integral Solutions - Integral Six
Splitting Expected Share Price Into Its Component Parts

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In this white paper we will build a model to split expected share price at some future time t into its component parts. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

Assume that the owner of $100,000$ shares of ABC Company uses those shares as collateral on a non-recourse, three year, interest-only, \$1.5 million loan. If at the end of the loan term share price is less than \$15.00 (i.e. total share value = 100,000 shares x \$15.00 per share = \$1.5 million) then the borrower defaults.

We are tasked with structuring a loan guarantee product and to to that we need to know the answers to the following questions:

Scenario	Description	Question
1	End-of-term share price is less than \$15.00	What is the weighted-average share price?
2	End-of-term share price is greater than \$15.00	What is the weighted-average share price?

The model parameters for the task at hand are...

Table 1: Model Parameters

Symbol	Description	Amount
S_0	Share price at time zero (\$)	30.00
μ	Average annual return - mean (%)	12.00
σ	Average annual return - volatility (%)	35.00
T	Loan term in years (#)	3.00

We will define the variable S_T to be random share price at time T, the variable $A(D)$ to be weighted-average share price at time T given that share price is less than \$15.00, and the variable $A(U)$ to be weighted-average share price at time T given that share price is greater than \$15.00. To answer the questions above we need to solve the following equation...

$$\mathbb{E}\Bigg[S_T\Bigg] = \text{Prob}\Bigg[\text{share price} < 15.00\Bigg] \times A(D) + \text{Prob}\Bigg[\text{share price} > 15.00\Bigg] \times A(U) \tag{1}$$

Expected Share Price

We will define the variable θ to be a random rate of return drawn from a normal distribution with mean m and variance v. Using the data in Table 1 above the equation for random share price at time T is...

$$S_T = S_0 \,\text{Exp}\Bigg\{\theta\Bigg\} \text{ ...where... } \theta \sim N\Bigg[m, v\Bigg] \text{ ...and... } m = \Bigg(\mu - \frac{1}{2}\sigma^2\Bigg)T \text{ ...and... } v = \sigma^2 T \tag{2}$$

We will define the variable x to be the value of the random return in Equation (2) above such that share price at time T equals some threshold value V_T. For our purposes the variable x will be the default point and the threshold value will be \$15.00 per share. The equation for the default point is...

$$\text{if... } S_0 \,\text{Exp}\Bigg\{x\Bigg\} = V_T \text{ ...then... } x = \ln\Bigg(\frac{V_T}{S_0}\Bigg) \tag{3}$$

We will define the function PDF(θ) to be the probability density function of a normally-distributed random variable θ with mean m and variance v. The equation for the probability density function is... [1]

$$\text{PDF}(\theta) = \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\} \text{ ...where... } \theta \sim N\Big[m, v\Big] \tag{4}$$

We will define the function CNDF(z, mean, variance) to be the cumulative normal distribution function. This function gives us the probability that the random return θ pulled from a normal distribution with mean m and variance v will be less than z. Using Equation (4) above the equation for the cumulative normal distribution function is... [1]

$$\text{CNDF}(z, m, v) = \text{Prob}\Big[\theta < z\Big] = \int\limits_{\infty}^{z} \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\}\delta\theta \text{ ...where... } \theta \sim N\Big[m, v\Big] \tag{5}$$

Note that the Excel version of the cumulative normal distribution function in Equation (5) above is...

$$\text{CNDF}(z, m, v) = NORM.DIST(z, m, \sqrt{v}, TRUE) \tag{6}$$

Using Equations (2), (3) and (4) above we can make the following statement...

$$\text{if... } \mathbb{E}\Big[S_T \Big| \theta < x\Big] = S_0 \int\limits_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\}\text{Exp}\Big\{\theta\Big\}\,\delta\theta \tag{7}$$

$$\text{and... } \mathbb{E}\Big[S_T \Big| \theta > x\Big] = S_0 \int\limits_{x}^{\infty} \sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\}\text{Exp}\Big\{\theta\Big\}\,\delta\theta \tag{8}$$

$$\text{then... } \mathbb{E}\Big[S_T\Big] = \mathbb{E}\Big[S_T \Big| \theta < x\Big] + \mathbb{E}\Big[S_T \Big| \theta > x\Big] \tag{9}$$

Using Appendix Equation (25) below and Equation (6) above the solutions to Equations (7) and (8) above are...

$$\mathbb{E}\Big[S_T \Big| \theta < x\Big] = S_0\,\text{Exp}\left\{m + \frac{1}{2}\,v\right\}\text{CNDF}(x, m + v, v) \tag{10}$$

$$\mathbb{E}\Big[S_T \Big| \theta > x\Big] = S_0\,\text{Exp}\left\{m + \frac{1}{2}\,v\right\}\Big(1 - \text{CNDF}(x, m + v, v)\Big) \tag{11}$$

Weighted-Average Share Price

Using Equations (10) and (11) above we will define the variables $A(D)$ and $A(U)$ in Equation (1) above as follows...

$$A(D) = \mathbb{E}\Big[S_T \Big| \theta < x\Big] \div \text{Prob}\Big[\theta < x\Big] \text{ ...and... } A(U) = \mathbb{E}\Big[S_T \Big| \theta > x\Big] \div \text{Prob}\Big[\theta > x\Big] \tag{12}$$

Using Equation (12) above we will rewrite Equation (1) above as...

$$\mathbb{E}\Big[S_T\Big] = \text{Prob}\Big[\theta < x\Big] \times A(D) + \text{Prob}\Big[\theta > x\Big] \times A(U) \tag{13}$$

The Answers To Our Hypothetical Problem

Using Equation (3) above and the data in Table 1 above the equation for our default point x is...

$$x = \ln\left(\frac{15.00}{30.00}\right) = -0.6931 \tag{14}$$

Using Equation (2) above and the data in Table 1 above the equations for our mean and variance are...

$$m = \left(0.12 - \frac{1}{2} \times 0.35^2\right) \times 3.00 = 0.1763 \text{ ...and... } v = 0.35^2 \times 3.00 = 0.3675 \tag{15}$$

Using Equations (10), (14) and (15) above and the data in Table 1 above the equation for expected share price given that share price is below the default point is...

$$\mathbb{E}\bigg[S_3\,\bigg|\,\theta < -0.6931\bigg] = 30.00 \times \mathrm{Exp}\left\{0.1763 + \frac{1}{2} \times 0.3675\right\} \times 0.0207 = \$0.89$$
$$\text{given that } \ NORM.DIST(-0.6931, 0.1763 + 0.3675, \sqrt{0.3675}, TRUE) = 0.0207 \tag{16}$$

Using Equations (11), (14) and (15) above and the data in Table 1 above the equation for expected share price given that share price is below the default point is...

$$\mathbb{E}\bigg[S_3\,\bigg|\,\theta > -0.6931\bigg] = 30.00 \times \mathrm{Exp}\left\{0.1763 + \frac{1}{2} \times 0.3675\right\} \times (1 - 0.0207) = \$42.11$$
$$\text{given that } \ 1 - NORM.DIST(-0.6931, 0.1763 + 0.3675, \sqrt{0.3675}, TRUE) = 0.9793 \tag{17}$$

Using Equations (5) and (6) above and the data in Table 1 above the equation for the probability that the random return will be less than the default point is...

$$\mathrm{Prob}\bigg[\theta < -0.6931\bigg] = NORM.DIST(-0.6931, 0.1763, \sqrt{0.3675}, TRUE) = 0.0758 \tag{18}$$

Using Equations (5) and (6) above and the data in Table 1 above the equation for the probability that the random return will be greater than the default point is...

$$\mathrm{Prob}\bigg[\theta > -0.6931\bigg] = 1 - NORM.DIST(-0.6931, 0.1763, \sqrt{0.3675}, TRUE) = 0.9242 \tag{19}$$

Using Equation (12), (16), (17), (18) and (19) above the answer to our problem is...

$$A(D) = \$0.89 \div 0.0758 = \$11.72 \ \text{ ...and... } \ A(U) = \$42.11 \div 0.9242 = \$45.56 \tag{20}$$

To prove that the answers in Equation (20) are correct we will use the average return mean (μ) in Table 1 above and make sure that Equation (13) above holds...

$$\$30.00 \times \mathrm{Exp}\left\{0.12 \times 3.00\right\} = \$43.00 = \bigg(0.0758 \times \$11.72\bigg) + \bigg(0.9242 \times \$45.56\bigg) \tag{21}$$

Appendix

A. The solution to the following integral is...

$$\begin{aligned} I &= \int_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\}\mathrm{Exp}\left\{\theta\right\}\delta\theta \\ &= \int_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta^2 - 2\,m\,\theta + m^2 - 2\,v\,\theta)\right\}\delta\theta \\ &= \int_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta^2 - 2\,m\,\theta + m^2 - 2\,v\,\theta + 2\,m\,v - 2\,m\,v + v^2 - v^2)\right\}\delta\theta \end{aligned} \tag{22}$$

We will make the following definition...

$$(\theta - (m + v))^2 = \theta^2 - 2\,m\,\theta + m^2 - 2\,v\,\theta + 2\,m\,v + v^2 \tag{23}$$

Using the definition in Equation (23) above we can rewrite Equation (22) above as...

$$\begin{aligned}
I &= \int\limits_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta-(m+v))^2)-\frac{1}{2\,v}\,(-2\,m\,v-v^2)\right\}\delta\theta \\
&= \int\limits_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta-(m+v))^2)\right\}\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(-2\,m\,v-v^2)\right\}\delta\theta \\
&= \mathrm{Exp}\left\{-\frac{1}{2\,v}\,(-2\,m\,v-v^2)\right\}\int\limits_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta-(m+v))^2)\right\}\delta\theta \\
&= \mathrm{Exp}\left\{m+\frac{1}{2}\,v\right\}\int\limits_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta-(m+v))^2)\right\}\delta\theta \qquad (24)
\end{aligned}$$

Using Equation (5) above we can rewrite Equation (24) above as...

$$I = \mathrm{Exp}\left\{0.12\right\}\mathrm{CNDF}(x, m+v, v) \qquad (25)$$

References

[1] Gary Schurman, *The Calculus of the Normal Distribution*, October, 2010.